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NOTES ON BERGMAN PROJECTION TYPE OPERATOR RELATED WITH BESOV SPACE

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Abstract. Let Qf be the maximal derivative of f with respect to the Bergman metric b_B. In this paper, we will find conditions such that $(1-\parallel z \parallel)^s (Qf)^p(z)$ is bounded on B. We will also find conditions such that Bergman projection type operator P_r is bounded operator from $L^p(B, d\mu_r)$ to the holomorphic Besov p-space $\mathcal{B}_p^s(B)$ with weight s.

1. Introduction

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $\parallel z \parallel^2 = \langle z, z \rangle$.

Let Ω be any bounded domain in $\mathbb{C}^n$. Let $f \in C^1(\Omega)$ and $\xi \in \mathbb{C}^n$. The maximal derivative of f with respect to the Bergman metric b_Ω is defined by

$$\hat{Q}f(z) = \sup_{\|\xi\|=1} \frac{|\langle df(z), \xi \rangle|}{b_\Omega(z, \xi)}, \ z \in \Omega$$

where

$$\langle df(z), \xi \rangle = \sum_{i=1}^n \left[\frac{\partial f(z)}{\partial z_i} \xi_i + \frac{\partial f(z)}{\partial \overline{z}_i} \overline{\xi}_i \right].$$

If $f \in H(\Omega)$ where $H(\Omega)$ is the set of holomorphic functions on Ω, then the quantity $\hat{Q}f$ is reduced to

$$Qf(z) = \sup_{\|\xi\|=1} \frac{|\langle \nabla f(z), \xi \rangle|}{b_\Omega(z, \xi)}, \quad z \in \Omega, \quad \xi \in \mathbb{C}^n$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of f.

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For Lebesgue measure ν in $\mathbb{C}^n$, let $d\lambda(z) = K(z,z)d\nu(z)$ where $K(z,w)$ is Bergman kernel. Let $\delta_\Omega(z)$ be the Euclidean distance from z to the boundary $\partial\Omega$. For $0 < p < \infty$ and $s \in \mathbb{R}$, the holomorphic Besov p-sapce $\mathcal{B}_p^s(\Omega)$ with weight s is defined by the space of all holomorphic functions f on Ω such that

$$\| f \|_{p,s} = \left\{ \int_\Omega (Qf)^p(z)\delta_\Omega(z)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$

In this paper, we will consider the case where Ω is open unit ball in $\mathbb{C}^n$. Let B be the open unit ball in the complex space $\mathbb{C}^n$ and S the boundary of B. For $z \in B, \xi \in \mathbb{C}^n$, the Bergman metric(on B) $b_B : B \times \mathbb{C}^n \longrightarrow R$ is given by

$${b_B}^2(z,\xi) = \frac{n+1}{(1- \| z \|^2)^2}[\,(1 - \| z \|^2)\| \xi \|^2 + |\langle z, \xi\rangle|^2\,]\ .$$

The quantity Qf for the unit ball B is invariant under the group $Aut(B)$ of holomorphic automorphisms of B. Namely, $Q(f \circ \varphi) = (Qf) \circ \varphi$ for all $\varphi \in Aut(B)$.

Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. The Bergman space $L_a^2(B, d\nu)$ is defined to be the subspace of $L^2(B, d\nu)$ consisting of analytic functions.

Fix a point $z \in B$. Since the functional e_z given by $e_z(f) = f(z), f \in L_a^2(B, d\nu)$, is continuous, there exists a function $K(\cdot, z) \in L_a^2(B, d\nu)$ such that

$$f(z) = \int_B f(w)\overline{K(w,z)}d\nu(w)$$

by the Riesz representation theorem. The function $K(z,w)$ is called the Bergman reproducing kernel in $L_a^2(B, d\nu)$. It is well known that $K(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1}}$(See [9]).

Let $0 < p < \infty$ and $s \in \mathbb{R}$. The holomorphic Besov p-spaces $\mathcal{B}_p^s(B)$ with weight s is defined by the space of all holomorphic functions f on the unit ball B such that

$$\| f \|_{p,s} = \left\{ \int_B (Qf)^p(z)(1- \| z \|^2)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$

Here $d\lambda(z) = K(z,z)d\nu(z) = (1- \| z \|^2)^{-n-1}d\nu(z)$ is an invariant volume measure with respect to the Bergman metric on B.

For a fixed $p \in (0, \infty)$, $\mathcal{B}_p^s(B)$ is an increasing family of function spaces in s; that is, if $-\infty < s \leq t < +\infty$, then $\mathcal{B}_p^s(B) \subset \mathcal{B}_p^t(B)$. The holomorphic Besov p-space $\mathcal{B}_p^s(B)$ with weight s include many well known spaces as special case. $\mathcal{B}_p^s(B)$ is the usual Hardy space $H^p(B)$ for $s = n$, the Bergman space $L_a^p(B)$ for $s = n+1$. In particular, the diagonal Besov space $\mathcal{B}_p^0(B)$ are shown to be the Möbius invariant subsets of the Bloch space(See [3]).

In recent years, there have been many papers focused on studying the Besov space and it's applications(See [4],[6],[7] and [10]).

In section 2, we will find conditions such that $(1- \parallel z \parallel)^s (Qf)^p(z)$ is bounded on B.

The orthogonal projection operator P from $L^2(B, d\nu)$ to $L_a^2(B, d\nu)$ is denoted by

$$Pf(z) = \int_B f(w)K(z,w)d\nu(w).$$

P is called the Bergman projection. The Bergman projection is used in many areas related with Hankel operators and Toeplitz operators(See [1],[8],[11] and [12]).

The measure μ_r is the weighted Lebesgue measure:

$$d\mu_r(z) = c_r(1- \parallel z \parallel^2)^r d\nu(z)$$

where $r > -1$ is fixed, and c_r is a normalization constant such that $\mu_r(B) = 1$. Define the Bergman projection type operator P_r by

$$P_r f(z) = \int_B \frac{f(w)}{(1-\langle z, w\rangle)^{n+r+1}} d\mu_r(w).$$

In section 3, we will find conditions such that P_r is bounded operator from $L^p(B, d\mu_r)$ to the holomorphic Besov p-spaces $\mathcal{B}_p^s(B)$ with weight s.

2. Holomorphic Besov p-space $\mathcal{B}_p^s(B)$ with weight s

The traditional holomorphic Besov space $\mathcal{B}_p(\Omega)$ is a subspace of $L_a^2(\Omega)$ with semi-norm

$$\parallel f \parallel_{\mathcal{B}_p} = \left\{ \int_\Omega (\nabla f)^p(z) \delta_\Omega(z)^p d\lambda(z) \right\}^{\frac{1}{p}} < \infty$$

where $\delta_\Omega(z)/2$ is the distance from z to $\partial\Omega$. It is known that the fact $\int_\Omega \delta_\Omega(z)^{-q} d\nu(z) = \infty$ when $q \geq 1$ implies that $\mathcal{B}_p(\Omega) = \mathbb{C}$ when $p \leq n$ and Ω is a smoothly bounded strictly pseudo convex domain in $\mathbb{C}^n$.

If Ω is the unit ball B in $\mathbb{C}^n$ and ν is the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(\Omega) = 1$, we can find the following result.

THEOREM 2.1. *Let $n \geq 2$ and $0 < p \leq 2n$. If $f \in H(B)$ and*

$$\int_B (Qf)^p(z)d\lambda(z) < \infty,$$

then f is constant.

Proof. See [3], Lemma 2.11. □

These results show that the above semi-norm is not natural when $p \leq n$. In this paper, we will consider the holomorphic Besov p-space $\mathcal{B}_p^s(B)$ with weight s.

Let $a \in B$ and let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a, which is given by $P_0 = 0$, and

$$P_a z = \frac{\langle z, a\rangle}{\langle a, a\rangle} a, \quad if \quad a \neq 0.$$

Let $Q_a = I - P_a$. Define φ_a on B by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - \| a \|^2} Q_a z}{1 - \langle z, a\rangle}.$$

THEOREM 2.2. *For every $a \in B$, φ_a has the following properties:*

(i) *The identity*

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1 - \| a \|^2)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)}$$

holds for all $z \in \overline{B}, w \in \overline{B}$.

(ii) *The identity*

$$1 - \| \varphi_a(z) \|^2 = \frac{(1 - \| a \|^2)(1 - \| z \|^2)}{|1 - \langle z, a\rangle|^2}$$

holds for every $z \in \overline{B}$.

(iii) *φ_a is a homeomorphism of $\overline{B}$ onto $\overline{B}$.*

Proof. See [9], Theorem 2.2.2. □

THEOREM 2.3. *For $z \in B$, c is real, $t > -1$, define*

$$I_{c,t}(z) = \int_B \frac{(1 - \| w \|^2)^t}{|1 - \langle z, w\rangle|^{n+1+c+t}} d\nu(w).$$

Then,

(i) *$I_{c,t}(z)$ is bounded in B if $c < 0$;*

(ii) *$I_{0,t}(z) \approx -\log(1 - \| z \|^2)$ as $\| z \| \to 1^-$;*

(iii) *$I_{c,t}(z) \approx (1 - \| z \|^2)^{-c}$ as $\| z \| \to 1^-$ if $c > 0$.*

Proof. See [9], Proposition 1.4.10. $\square$

LEMMA 2.4. *If f is holomorphic and $\frac{Qf(w)}{1-\|w\|^2}$ is Lebesgue integrable on B, then*

$$Qf(0) \leq (n+1)\int_B Qf(w)(1-\| w \|^2)^n d\lambda(w).$$

Proof. By the definition of Bergman metric,

$$\begin{aligned} {b_B}^2(z,\xi) &= (n+1)\frac{(1-\| z \|^2)\| \xi \|^2 + |\langle z,\xi\rangle|^2}{(1-\| z \|^2)^2} \\ &\leq (n+1)\frac{(1-\| z \|^2)\| \xi \|^2 + \| z \|^2\| \xi \|^2}{(1-\| z \|^2)^2} \\ &\leq (n+1)\frac{\| \xi \|^2}{(1-\| z \|^2)^2}. \end{aligned}$$

By the mean value theorem,

$$f(t\eta) = \int_B f\circ\varphi_{t\eta}(w)d\nu(w)$$

for $f\in H(B), \eta\in B$ and $t\in[0,1]$.

$$\begin{aligned} |\langle\nabla f(0),\eta\rangle| &= \left|\int_B \langle\nabla f(-w), \left[\frac{d}{dt}\varphi_{t\eta}(w)\right]_{t=0}\rangle d\nu(w)\right| \\ &= \left|\int_B \langle\nabla f(-w), \eta-\langle w,\eta\rangle w\rangle d\nu(w)\right| \\ &\leq \int_B \frac{\left|\langle\nabla f(-w), \frac{\eta-\langle w,\eta\rangle w}{\|\eta-\langle w,\eta\rangle w\|}\rangle\right|}{b_B\left(-w, \frac{\eta-\langle w,\eta\rangle w}{\|\eta-\langle w,\eta\rangle w\|}\right)} b_B(-w,\eta-\langle w,\eta\rangle w)d\nu(w) \\ &\leq (n+1)\int_B Qf(w)b_B(-w,\eta-\langle w,\eta\rangle w)d\nu(w) \\ &\leq (n+1)\int_B \frac{Qf(w)}{1-\| w \|^2}d\nu(w) \\ &\leq (n+1)\int_B (1-\| w \|^2)^n Qf(w)d\lambda(w). \end{aligned}$$

Thus,

$$Qf(0) \leq (n+1)\int_B Qf(w)(1-\| w \|^2)^n d\lambda(w).$$

$\square$

THEOREM 2.5. *Let $1 < p < \infty$. If s is a real number such that $-np < s < n$ and $\| f \|_{p,s} < \infty$, then $(1- \| z \|^2)^s (Qf)^p(z)$ is bounded on B.*

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ where $q > 1$. If $s < n$, then $t = (n-s)q + s - n - 1 > -1$. If $-np < s$, then $nq + sq - s > 0$. By Theorem 2.3,

$$\begin{aligned}
&\left(\int_B \frac{(1- \| \xi \|^2)^{(n-s)q}}{|1 - \langle z, \xi \rangle|^{2nq}} (1- \| \xi \|^2)^s d\lambda(\xi) \right)^{1/q} \\
&= \left(\int_B \frac{(1- \| \xi \|^2)^t}{|1 - \langle z, \xi \rangle|^{n+1+t+(nq+sq-s)}} d\nu(\xi) \right)^{1/q} \\
&\approx (1- \| z \|^2)^{-n-s+s/q} \\
&\approx (1- \| z \|^2)^{-n-s/p}.
\end{aligned}$$

By Lemma 2.4,

$$Qf(0) \leq (n+1) \int_B Qf(w)(1 - \| w \|^2)^n d\lambda(w).$$

Put $\xi = \varphi_z(w)$. By Theorem 2.2,

$$\begin{aligned}
Qf(z) &= Q(f \circ \varphi_z)(0) \\
&\leq (n+1) \int_B Q(f \circ \varphi_z)(w)(1 - \| w \|^2)^n d\lambda(w) \\
&\leq (n+1) \int_B Qf(\xi)(1 - \| \varphi_z(\xi) \|^2)^n d\lambda(\xi) \\
&\leq (n+1) \int_B Qf(\xi) \frac{(1- \| z \|^2)^n (1- \| \xi \|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}} d\lambda(\xi) \\
&\leq (n+1)(1- \| z \|^2)^n \left(\int_B (Qf)^p(\xi)(1- \| \xi \|^2)^s d\lambda(\xi) \right)^{1/p} \\
&\qquad \left(\int_B \frac{(1- \| \xi \|^2)^{(n-s)q}}{|1 - \langle z, \xi \rangle|^{2nq}} (1- \| \xi \|^2)^s d\lambda(\xi) \right)^{1/q}
\end{aligned}$$

where the last inequality follows from Hölder inequality for $\frac{1}{p} + \frac{1}{q} = 1$. This implies that

$$Qf(z) \leq C(1- \| z \|^2)^{-s/p} \| f \|_{p,s}$$

for some constant C. This shows that if s is a real number such that $-np < s < n$ and $\| f \|_{p,s} < \infty$, then $(1- \| z \|^2)^s (Qf)^p(z)$ is bounded on B. $\square$

3. Bounded Bergman projection type operator related with Besov space

In [10], Timoney showed that the linear space of all holomorphic function $f : B \longrightarrow \mathbb{C}$ which satisfy

$$\sup_{z\in B}(1- \| z \|^2) \| \nabla f(z) \| < \infty$$

is equivalent to the space of all holomorphic function which satisfy

$$\sup_{z\in B} Qf(z) < \infty.$$

THEOREM 3.1. *Let $p > 2n$ and $s > n$. Then for every $f \in H(B)$,*

$$\int_B (Qf)^p(z)(1- \| z \|^2)^s d\lambda(z) \approx \int_B \| \nabla f(z) \|^p (1- \| z \|^2)^{p+s} d\lambda(z)$$

Proof. See [3], Lemma 2.8. □

Let $L^2_{a,r} = L^2_a(B, d\mu_r)$ be the subspace of $L^2(B, d\mu_r)$ consisting of analytic functions. If we equip $L^2_{a,r}$ with the norm $\| f \|_{2,r} = \sqrt{\int_B |f|^2 d\mu_r}$, then $L^2_{a,r}$ is a Banach space for each $r > -1$.

Fix a point $z \in B$. Since the functional e_z given by $e_z(f) = f(z), f \in L^2_{a,r}$, is continuous, there exists a function $k_{r,z} \in L^2_{a,r}$ such that

$$f(z) = \int_B f(w)\overline{k_{r,z}(w)} d\mu_r(w)$$

by the Riesz representation theorem. The function $K_r(z, w) = \overline{k_{r,z}(w)}$ is called the weighted Bergman kernel. Also it is well known that

$$K_r(z, w) = \frac{1}{(1 - \langle z, w\rangle)^{r+n+1}}$$

(See [9]). It was shown in [5] that if $f \in L^1_{a,r}, r > -1$, then

$$f(z) = \int_B \frac{f(w)}{(1 - \langle z, w\rangle)^{n+r+1}} d\mu_r(w).$$

Suppose $1 \le p < +\infty$ and $r > 0$. Let $L^p_{a,r}$ be the subspace of $L^p(B, d\mu_r)$ consisting of analytic functions. Define Bergman projection type operator P_r by

$$P_r f(z) = \int_B \frac{f(w)}{(1 - \langle z, w\rangle)^{n+r+1}} d\mu_r(w).$$

Since $P_r f = f$ for all analytic f in $L^1(B, d\mu_r)$, P_r is a projection from $L^1(B, d\mu_r)$ onto $L^1_a(B, d\mu_r)$.

In [2], the author proved that P_r is a bounded projection operator from $L^p(B, d\nu)$ onto $L^p_a(B, d\nu)$.

In the proof of Theorem 3.2, we will use $C_{n,r}$ to denote constant depending only on n and r, but it is not always the same at each appearance.

THEOREM 3.2. *Let $p > 2n$ and $r > 0$. If $f \in L^p(B, d\mu_r)$, then*

$$\| P_r f \|_{p,s} \leq C_{n,r} \| f \|_{L^p(B,d\mu_r)}$$

for $s > 2n + r + 1$.

Proof. Differentiating under the integral sign, we obtain

$$\frac{\partial}{\partial z_j}(P_r f)(z) = (n + r + 1) \int_B \frac{f(w)(-\overline{w_j})}{(1 - \langle z, w\rangle)^{n+r+2}} d\mu_r(w)$$

for $j = 1, 2, \cdots, n$. This shows that

$$\| \nabla P_r f(z) \| \leq C_{n,r} \int_B \frac{|f(w)|}{|1 - \langle z, w\rangle|^{n+r+2}} d\mu_r(w).$$

Let $\frac{1}{p} + \frac{1}{q} = 1$. By the Hölder inequality,

$$\begin{aligned}
&\| \nabla P_r f(z) \|^p \\
&\leq \left(C_{n,r} \int_B \frac{|f(w)|}{|1 - \langle z, w\rangle|^{n+r+2}} d\mu_r(w) \right)^p \\
&= C_{n,r} \int_B |f(w)|^p d\mu_r(w) \left(\int_B \frac{1}{|1 - \langle z, w\rangle|^{q(n+r+2)}} d\mu_r(w) \right)^{p/q}.
\end{aligned}$$

By Theorem 2.3,

$$\begin{aligned}
&\int_B \frac{1}{|1 - \langle z, w\rangle|^{q(n+r+2)}} d\mu_r(w) \\
&= c_r \int_B \frac{(1- \| w \|^2)^r}{|1 - \langle z, w\rangle|^{q(n+r+2)}} d\nu(w) \\
&= c_r \int_B \frac{(1- \| w \|^2)^r}{|1 - \langle z, w\rangle|^{n+1+r+1+(q-1)(n+r+2)}} d\nu(w) \\
&\approx (1- \| z \|^2)^{-1-(q-1)(n+r+2)}.
\end{aligned}$$

By Theorem 3.1,

$$
\begin{aligned}
\| P_r f \|_{p,s}^p &= \int_B (QP_r f)^p(z)(1- \| z \|^2)^s d\lambda(z) \\
&\approx \int_B (1 - \| z \|^2)^p \| \nabla P_r f(z) \|^p (1- \| z \|^2)^s d\lambda(z) \\
&\leq C_{n,r} \| f \|_{L^p(B,d\mu_r)}^p \int_B (1 - \| z \|^2)^p \\
&\quad \left(\int_B \frac{1}{|1 - \langle z, w \rangle|^{q(n+r+2)}} d\mu_r(w) \right)^{p/q} (1- \| z \|^2)^s d\lambda(z) \\
&\leq C_{n,r} \| f \|_{L^p(B,d\mu_r)}^p \\
&\quad \int_B (1 - \| z \|^2)^{p+s-n-1-(p/q)(1+(q-1)(n+r+2))} d\nu(z).
\end{aligned}
$$

If $s > n - p + (p/q)(1 + (q-1)(n + r + 2)) = 2n + r + 1$, then

$$
\| P_r f \|_{p,s} \leq C_{n,r} \| f \|_{L^p(B,d\mu_r)} .
$$

□

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